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The local fractional Hilbert transform in fractal space

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Abstract. In this paper, we establish local fractional Hilbert transform in fractal space, consider some properties of local fractional Hilbert Transforms.

Keywords: fractal space, local fractional Hilbert transform , local fractional derivative.

1. Introduction

In the past ten years, Local fractional calculus[1-26] has been widely applied to many fields such as mathematics , image processing and signal processing etc .Some authors have given many definitions of local fractional derivatives and local fractional integrals (also called fractal calculus) [1-16]. Hereby we rewrite the following local fractional derivative which is given by [15,16,25]

$$f^{(\alpha)}(x_0)=\frac{df(x)}{dx^{\alpha}}|_{x=x_0}=\lim_{\delta x\to 0}\frac{\Delta^{\alpha}(f(x)-f(x_0))}{(x-x_0)^{\alpha}}, \quad \text{for } 0<\alpha\leq 1, \tag{1.1}$$

where

$$\Delta^{\alpha}(f(x)-f(x_0))\cong\Gamma(1+\alpha)\Delta(f(x)-f(x_0)),$$

and local fractional integral of $f(x)$ defined by [15-16]

$${}_aI_b^{(\alpha)}f(t)=\frac{1}{\Gamma(1+\alpha)}\int_a^b f(t)(dt)^{\alpha}=\frac{1}{\Gamma(1+\alpha)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f(t_j)(\Delta t_j)^{\alpha}, \tag{1.2}$$

with $\Delta t_j=t_{j+1}-t_j$ and $\Delta t=\max\{\Delta t_1,\Delta t_2,\ldots,\Delta t_j,\ldots\}$, where for $j=1,2,\ldots,N-1$, $[t_j,t_{j+1}]$ is a partition of the interval $[a,b]$ and $t_0=a$, $t_N=b$.

The aim of this paper is to define local fractional Hilbert transforms based on Local fractional calculus. This paper is organized as follows. In section 2, local fractional Hilbert transforms is defined; Section 3 presents Properties of local fractional Hilbert transforms.

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2. The local fractional Hilbert transform and some examples

In the section, both local fractional Hilbert transform and its inverse transform are defined.

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Definition 2.1 (The local fractional Hilbert transform). If $f(x)$ is defined on the real line $-\infty < x < \infty$, its local fractional Hilbert transform, denoted by $f_x^{H,\alpha}(x)$ is defined by

$$H_\alpha\{f(t)\} = \hat{f}_H^\alpha(x) = \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(t)}{(t-x)^\alpha}(dt)^\alpha \tag{2.1}$$

where x is real and the integral is treated as a Cauchy principal value, that is,

$$\begin{aligned}&\frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(t)}{(t-x)^\alpha}(dt)^\alpha \\ &= \lim_{\varepsilon\to 0}\left[\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{x-\varepsilon}\frac{f(t)}{(t-x)^\alpha}(dt)^\alpha + \frac{1}{\Gamma(1+\alpha)}\int_{x+\varepsilon}^{\infty}\frac{f(t)}{(t-x)^\alpha}(dt)^\alpha\right]\end{aligned} \tag{2.2}$$

To obtain the inverse local fractional Hilbert transform, write again (2.1) as

$$\begin{aligned}&\hat{f}_H^\alpha(x) = \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{f(t)}{(t-x)^\alpha}(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f(t)g(x-t)(dt)^\alpha = f(x)*g(x),\end{aligned} \tag{2.3}$$

where $g(x) = -\frac{1}{x^\alpha}$, Application of the local fractional Yang-Fourier transform [15,16] with respect to x gives

$$f_\omega^{F,\alpha}(\omega) = \frac{f_{H,\omega}^{F,\alpha}(\omega)}{g_\omega^{F,\alpha}(\omega)},\ g_\omega^{F,\alpha}(\omega) = i^\alpha \pi^\alpha \operatorname{sgn}_\alpha \omega^\alpha \tag{2.4}$$

Taking the inverse Yang-Fourier transform [15,16], we find the solution for $f(x)$ as

$$f(x) = \frac{1}{(2\pi)^\alpha}\int_{-\infty}^{\infty} i^\alpha \pi^\alpha \operatorname{sgn}_\alpha \omega^\alpha f_{H,\omega}^{F,\alpha}(\omega) E_\alpha(i^\alpha \omega^\alpha x^\alpha)(d\omega)^\alpha$$

which is, by the Convolution Theorem [15,16]

$$f(x) = \frac{1}{(2\pi)^\alpha}\oint_R \frac{\hat{f}_H^\alpha(\xi)}{(x-\xi)^\alpha}(d\xi)^\alpha = -\frac{1}{(2\pi)^\alpha}\oint_R \frac{\hat{f}_H^\alpha(\xi)}{(\xi-x)^\alpha}(d\xi)^\alpha = -H\{\hat{f}_H^\alpha(\xi)\} \tag{2.5}$$

Obviously, $-H_\alpha^2\{f(t)\} = -H_\alpha[H_\alpha\{f(t)\}] = f(x)$ and hence, $H_\alpha^{-1} = -H_\alpha$. Thus, the inverse local fractional Hilbert transform is given by

$$f(t) = H_\alpha^{-1}\{\hat{f}_H^\alpha(x)\} = -H_\alpha\{\hat{f}_H^\alpha(x)\} = -\frac{1}{(2\pi)^\alpha}\oint_R \frac{\hat{f}_H^\alpha(\xi)}{(x-\xi)^\alpha}(d\xi)^\alpha \tag{2.6}$$

Example 2.1 Find the local fractional Hilbert transform of

$$f(t) = \frac{t^\alpha}{t^{2\alpha}+a^{2\alpha}},\quad a>0$$

We get, by definition,

$$\hat{f}_H^\alpha(x)=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{t^\alpha}{(t^{2\alpha}+a^{2\alpha})(t-x)^\alpha}(dt)^\alpha$$
$$=\frac{1}{(a^{2\alpha}+x^{2\alpha})\Gamma(1+\alpha)}\int_{-\infty}^{\infty}[\frac{a^{2\alpha}}{(t^{2\alpha}+a^{2\alpha})}+\frac{x^\alpha}{(t-x)^\alpha}-\frac{x^\alpha t^\alpha}{(t^{2\alpha}+a^{2\alpha})}](dt)^\alpha$$
$$=\frac{1}{(a^{2\alpha}+x^{2\alpha})\Gamma(1+\alpha)}[a^{2\alpha}\int_{-\infty}^{\infty}\frac{1}{(t^{2\alpha}+a^{2\alpha})}(dt)^\alpha+x^\alpha\int_{-\infty}^{\infty}\frac{1}{(t-x)^\alpha}(dt)^\alpha-x^\alpha\int_{-\infty}^{\infty}\frac{t^\alpha}{(t^{2\alpha}+a^{2\alpha})}(dt)^\alpha]$$

The second and third integrals as the Cauchy principal value vanish and hence, only the first integral makes a non-zero contribution. Thus, we have

$$\hat{f}_H^\alpha(x)=\frac{a^\alpha}{(a^{2\alpha}+x^{2\alpha})}\arctan_\alpha\frac{t^\alpha}{a^\alpha}\Big|_{-\infty}^{\infty}=\frac{\pi^\alpha a^\alpha}{(a^{2\alpha}+x^{2\alpha})} \qquad (2.7)$$

Example 2.2 Find the local fractional Hilbert transform of

(a) $f(t)=\cos_\alpha \omega^\alpha t^\alpha$ and (b) $f(t)=\sin_\alpha \omega^\alpha t^\alpha$

$$\hat{f}_H^\alpha(x)=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha t^\alpha}{(t-x)^\alpha}(dt)^\alpha$$
$$=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha[\omega^\alpha(t-x)^\alpha+\omega^\alpha x^\alpha]}{(t-x)^\alpha}(dt)^\alpha$$
$$=\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha(t-x)^\alpha\cos_\alpha \omega^\alpha x^\alpha-\sin_\alpha \omega^\alpha(t-x)^\alpha\sin_\alpha \omega^\alpha x^\alpha}{(t-x)^\alpha}(dt)^\alpha$$
$$=\cos_\alpha \omega^\alpha x^\alpha\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha(t-x)^\alpha}{(t-x)^\alpha}(dt)^\alpha-\sin_\alpha \omega^\alpha x^\alpha\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\sin_\alpha \omega^\alpha(t-x)^\alpha}{(t-x)^\alpha}(dt)^\alpha$$

which is, the new variable $T=t-x$,

$$\hat{f}_H^\alpha(x)=\cos_\alpha \omega^\alpha x^\alpha\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\cos_\alpha \omega^\alpha T^\alpha}{T^\alpha}(dT)^\alpha$$
$$-\sin_\alpha \omega^\alpha x^\alpha\frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty}\frac{\sin_\alpha \omega^\alpha T^\alpha}{T^\alpha}(dT)^\alpha \qquad (2.8)$$

Obviously, the first integral vanishes because its integrand is an odd function of T. On the other hand, the second integral makes a non-zero contribution so that (2.8) gives

$$H_\alpha\{\cos_\alpha \omega^\alpha t^\alpha\}=-\pi^\alpha\sin_\alpha \omega^\alpha x^\alpha \qquad (2.9)$$

Similarly, it can be shown that

$$H_\alpha\{\sin_\alpha \omega^\alpha t^\alpha\}=\pi^\alpha\cos_\alpha \omega^\alpha x^\alpha \qquad (2.10)$$

3. Basic Properties of the local fractional Hilbert Transforms

Theorem 3.1 If $H_\alpha\{f(t)\}=\hat{f}_H^\alpha(x)$, then the following properties hold:

(a) $H_\alpha\{f(t+a)\}=\hat{f}_H^\alpha(x+a)$, (3.1)

(b) $H_\alpha\{f(at)\}=\hat{f}_H^\alpha(ax)$, (3.2)

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(c) $H_\alpha\{f(-at)\} = -\hat{f}_H^\alpha(-ax)$, (3.3)

(d) $H_\alpha\{f^{(\alpha)}(t)\} = \frac{d^\alpha \hat{f}_H^\alpha(x)}{dx^\alpha}$, (3.4)

(e) $H_\alpha\{tf(t)\} = x^\alpha \hat{f}_H^\alpha(x) + \frac{1}{\Gamma(1+\alpha)}\int_{-\infty}^{\infty} f(t)(dt)^\alpha$, (3.5)

(f) $F_\alpha\{H_\alpha\{f(t)\}\} = -i^\alpha \operatorname{sgn}_\alpha \omega^\alpha F_\alpha\{f(x)\}$, (3.6)

(g) $\| H_\alpha\{f(t)\}\|_\alpha = \| f(t)\|_\alpha$ (3.7)

where $\| f(t)\|_\alpha = \sqrt{\langle f, f\rangle_\alpha}$ denotes the norm in $L_{2,\alpha}(\mathbb{R})$,

(h) $H_\alpha[f](x) = \hat{f}_H^\alpha(x), H_\alpha[\hat{f}_H^\alpha](x) = -f$ (Reciprocity relations), (3.8)

(i) (Parseval's formulas).

$\langle f, H_\alpha g\rangle_\alpha = \langle -H_\alpha f, g\rangle_\alpha$ and $\langle H_\alpha f, g\rangle_\alpha = \langle f, -H_\alpha g\rangle_\alpha$, (3.9)

Proof. (a) We obtain, by definition ,set $u = t + a$

$$
\begin{aligned}
H_\alpha\{f(t+a)\} &= \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(t+a)}{(t-x)^\alpha}(dt)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(u)}{(u-a-x)^\alpha}(du)^\alpha \\
&= \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(u)}{(u-(x+a))^\alpha}(du) = \hat{f}_H^\alpha(x+a)
\end{aligned}
$$

(b) We find, by definition ,set $u = at$

$$
H_\alpha\{f(at)\} = \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(at)}{(t-x)^\alpha}(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(u)}{(u-ax)^\alpha}(du)^\alpha = \hat{f}_H^\alpha(ax)
$$

Similarly, result (c) can be proved.

(d) We have, by definition

$$
\begin{aligned}
H_\alpha\{f^{(\alpha)}(t)\} &= \frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f^{(\alpha)}(t)}{(t-x)^\alpha}(dt)^\alpha \\
&= \frac{f(t)}{(t-x)^\alpha}\Big|_{-\infty}^{\infty} - \frac{1}{\Gamma(1+\alpha)}\oint_R f(t)[\frac{1}{(t-x)^\alpha}]^{(\alpha)}(dt)^\alpha \\
&= -\frac{\Gamma(1-\alpha)}{\Gamma(1-2\alpha)}\frac{1}{\Gamma(1+\alpha)}\oint_R \frac{f(t)}{(t-x)^{2\alpha}}(dt)^\alpha = \frac{d^\alpha \hat{f}_H^\alpha(x)}{dx^\alpha}
\end{aligned}
$$

Proofs of (e) – (i) are similar and consequently,we omit it

Theorem 3.2 If $f(t)$ is an even function of t, then, an alternative form of the local fractional Hilbert transform is

$$\hat{f}_H^{\alpha}(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \oint_R \frac{f(t)-f(x)}{t^{2\alpha}-x^{2\alpha}} (dt)^{\alpha} \tag{3.10}$$

Proof. As the Cauchy principal value, we obtain

$$\frac{1}{\Gamma(1+\alpha)} \oint_R \frac{1}{(t-x)^{\alpha}} (dt)^{\alpha} = 0$$

Consequently,

$$\begin{aligned} \hat{f}_H^{\alpha}(x) &= \frac{1}{\Gamma(1+\alpha)} \oint_R \frac{f(t)-f(x)}{t^{\alpha}-x^{\alpha}} (dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \oint_R \frac{t^{\alpha}(f(t)-f(x))}{t^{2\alpha}-x^{2\alpha}} (dt)^{\alpha} + \frac{x^{\alpha}}{\Gamma(1+\alpha)} \oint_R \frac{(f(t)-f(x))}{t^{2\alpha}-x^{2\alpha}} (dt)^{\alpha} \end{aligned} \tag{3.11}$$

Since $f(t)$ is an even function, the integrand of the first integral of (3.11) is an odd function; hence, the first integral vanishes, and (3.11) gives (3.10). Since result (2.3) discovers that the local fractional Hilbert transform can be written as a convolution transform, we state the following.

Theorem 3.3 If f and $g \in L_{1,\alpha}(\mathbb{R})$ are such that their local fractional Hilbert transforms are also in $L_{1,\alpha}(\mathbb{R})$,Then

$$H_{\alpha}(f * g)(x) = (H_{\alpha} f * g)(x) = (f * H_{\alpha} g)(x) \tag{3.12}$$

and

$$(f * g)(x) = -(H_{\alpha} f * H_{\alpha} g)(x). \tag{3.13}$$

Proof. We obtain, by definition, $\xi = t - y$

$$\begin{aligned} H_{\alpha}(f * g)(x) &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{1}{(t-x)^{\alpha}} [\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(y) g(t-y) (dy)^{\alpha}] (dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(y)(dy)^{\alpha} [\frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{g(\xi)}{(\xi-(x-y))^{\alpha}} (d\xi)^{\alpha}] \\ &= \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} f(y)(H_{\alpha} g)(x-y)(dy)^{\alpha} = (f * H_{\alpha} g)(x) \end{aligned}$$

Likewise, we can obtain the second result in (3.12).

To prove (3.13), we replace g by $H_{\alpha} g$ in (3.12) and then use $H_{\alpha}^2 g = -g$.

4. Summary

In present paper we give local fractional Hilbert transforms as follows:

$$H_{\alpha}\{f(t)\} = \hat{f}_H^{\alpha}(x) = \frac{1}{\Gamma(1+\alpha)} \int_{-\infty}^{\infty} \frac{f(t)}{(t-x)^{\alpha}} (dt)^{\alpha} \quad \text{for } 0 < \alpha \le 1 \tag{4.1}$$

and its inverse transform

$$f(t)=H_{\alpha}^{-1}\{\hat{f}_{H}^{\alpha}(x)\}=-H_{\alpha}\{\hat{f}_{H}^{\alpha}(x)\}=-\frac{1}{(2\pi)^{\alpha}}\int_{-\infty}^{\infty}\frac{\hat{f}_{H}^{\alpha}(\xi)}{(x-\xi)^{\alpha}}(d\xi)^{\alpha} \quad \text{for} \quad 0<\alpha\leq 1 \qquad (4.2)$$

The transforming functions are local fractional continuous. That is to say, it is fractal function defined on fractal sets. Hilbert transforms in integer space are the special case of fractal dimension $\alpha=1$. It is a tool to deal with differential equation with local fractional derivative.

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