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The Quantizations of Gaussian Curvature and Vacuum Einstein Equation at Planck Scale

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Abstract: We have derived the canonical guantization of Gaussian curvature at Planck scale, and which shows that the corresponding eigenvalues of Ganssian curvature and area element are proportional to each other. Next, we rewrite vacuum Einstein equation at Planck scale as eigenvalue equation, we have proven that a solution (a spacelike hypersurface) of this equation is not the fixed space of constant curvature, but should be a series of spin networks with different eigen curvatures. We have also derived the quantized Gauss-Bonnet formula and discussed its application.

Introduction

It is known that physical space is effectively granular at Planck scale, there is no spatical continuity at short scale. When the curvature becomes very large, of the order of $1/L_p^2 = \frac{c^3}{G\hbar}$, quantum effects of space should be considered [1–3]. There are a number of different approaches to quantum gravity. One natural avenue is using quantum Riemannian geometry in place of the classical differential geometry.

In this paper, by using the quantization of area element of 2-surface at Planck scale, we would research into the quantizations of Gaussian curvature and vacuum Einstein equation at Planck scale, and then we have discussed their some applications.

The Canonical Quantization of Gaussian Curvature at Planck Scale

A. Corresponding Eigenvalues K_v and δA_v are Proportional to Each Other

There are two basic field variables (Ashtekar's new variables) in the canonical theory: the connection $A_a^i(\tau)$ and its conjugate momentum $E_i^a(\tau)$, from which the aria element dA of 2-surface at Planck scale can be quantized.[1–3] We will use the canonical guantization of area element dA of 2-surface to derive the canonical guantization of Gaussian curvature K.

We have already known that the canonical guantization of the area element dA of 2- surface at Planck scale can be represented as [1–3]

$$d\hat{A}\psi_r = 8\pi r L_p^2 \sum_{\alpha} \delta^{(2)}(x, v_\alpha)\sqrt{j_\alpha(j_\alpha+1)}\, d^2x\psi_r, \tag{1}$$

Where the index α labels the vertexes on 2-surface, according to (1) we have the following eguation

$$\hat{K}\,\hat{dA}\psi_r = K8\pi r L_p^2 \sum_\alpha \delta^{(2)}(x,v_\alpha)\sqrt{j_\alpha(j_\alpha+1)}d^2x\psi_r, \quad (2)$$

where K is Gaussian curvature, by integrating both sides of (2), take $\delta A_{v_\alpha} = dA+\varepsilon,$ and take the limit as $\varepsilon \to 0,$ we consider the Gaussian curvature K_{v_α} of 2-surface $\delta s_{v_\alpha}, \delta s_{v_\alpha}$ is a region of 2-surface, and δs_{v_α} is the sufficiently small neighborhood of some vertex v_α. Thus we have

$$\int_0^{\delta A_{v_\alpha}} \hat{K}_{v_\alpha}\,\hat{dA}\psi_r = \int K_{v_\alpha}\{8\pi r L_p^2\sqrt{j_\alpha(j_\alpha+1)}\delta^{(2)}(x,v_\alpha)\}d^2x\psi_r, \quad (3)$$

Therefore, we have

$$\hat{K}_{v_\alpha}\,\delta\hat{A}_{v_\alpha}\,\psi_r = K_{v_\alpha}\{8\pi r L_p^2\sqrt{j_\alpha(j_\alpha+1)}\}\psi_r, \quad (4a)$$

We see in (4a) that the eigenvalues δA_{v_α} of operator $\delta\hat{A}_{v_\alpha}$ should be [1–3]

$$\delta A_{v_\alpha} = 8\pi r L_p^2\sqrt{j_\alpha(j_\alpha+1)}, \quad (4b)$$

It is known that Gaussian curvature K can be explained as following: The parallel displacement of a tangent vector around the boundary C of some 2-surface δs must have the corresponding intrinsic angular difference $\Delta\varphi \equiv \Delta(c_\alpha)$, and according to Gauss-Bonnet theorem there must be [9–10]

$$\Delta(c_\alpha) = \oint_c d\theta - \oint_c k_g(s)ds = \iint_0^{\delta A_{v_\alpha}} K_{v_\alpha} dA = K_{v_\alpha}\delta A_{v_\alpha}. \quad (5)$$

Taking limit, we have

$$K_{v_\alpha} = \lim_{c\to v}\Delta(c_\alpha)(\frac{1}{\delta A_{v_\alpha}}), \quad (6)$$

Which satisfies the definition of Gaussian curvature [9–10]. In eguation (5), where $k_g(s)$ is the geodesic curvature of the smooth curve C, which vanishes if C happens to be a geodesic polygon. K_{v_α} is Gaussian Curvature. θ is the direction angle between the unite tangent vector T of curve C and the positive direction of curvilinearcoordinates u. where R denotes the region of the area element δs of 2-surface, its interior point is the vertex v, and its boundary curve C is the smooth and closed curve with arc length $s \in [0,l]$ as parameter. $\omega(0)$ is the unite tangent vector at $a(0)$ point of C, and $\omega(s)$ denotes the parallel displacement of $\omega(0)$ around the boundary curve C. Eguation (5) shows that the intrinsic angle difference $\Delta\varphi = \varphi(l)-\varphi(0) \equiv \Delta(c_\alpha)$ denotes the total change of the direction angle of the unite tangent vector $\omega(0)$ in the parallel displacement process around the boundary curve C. $\varphi(0)$ is the direction angle between $\omega(0)$ at $a(0)$ point and the curvilinear coordinate u, $\varphi(l)$ is the direction angle between $\omega(l)$ and the curvilinear coordinate u, where $\omega(l)$ denotes the parallel displacement of $\omega(0)$ from $a(0)$ point to $a(l)$ point .

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Let us suppose δA_{v_α} is the area of any sufficiently small 2-surface δs_{v_α}, which contains only the neighborhood of a vertex v_α; and δs_{v_α} is so small that K_{v_α} can be considered to be its Gaussian curvature. Omiting limit symbol in (6),

$$K_{v_\alpha} = \Delta(c_\alpha)\left(\frac{1}{\delta A_{v_\alpha}}\right). \tag{7}$$

Inserting (4b) into (7), we obtain

$$K_{v_\alpha} = \Delta(c_\alpha)\left(\frac{1}{8\pi r L_p^2 \sqrt{j_\alpha(j_\alpha+1)}}\right). \tag{8}$$

We see in (7) and (8) that the Gaussian curvature K_{v_α} of any sufficiently small 2-surface δs_{v_α} at Planck scale should be quantizing, since the area element δA_{v_α} of δs_{v_α} is quantizing at Planck scale. Obviously,the eigenvalues of δA_{v_α} and K_{v_α} at Planck scale should be relating to half-integers j_α, which are the multiplets of half-intergers j_α, and the product of the corresponding eigenvalues of δA_{v_α} and K_{v_α} eguals the corresponding intrinsic angular difference $\Delta(c_\alpha)$, which is some determinate constant relating to the boundary condition C. Eguation (8) shows that the corresponding eigenvalnes of Gaussian curvature K_{v_α} and area element δA_{v_α} are proportional to each other. And the proportional constant $\Delta(c_\alpha)$ is the corresponding intrinsic angular difference, which is only relative to the path of the parallel displacement of the tangent vector around the boundary C; and which is not relative to the area δA_{v_α} of δs_{v_α}.

We will prove the point of view as mentioned above: By using the geodesic canonical coordinates,the Gaussian curvature can be represented as

$$K(u^1,u^2) = -\frac{1}{\sqrt{g_{11}}}\left(\frac{\partial^2\sqrt{g_{11}}}{\partial(u^2)^2}\right), \tag{9}$$

Taking the area integrations both sides of (10),we have

$$\iint_{\delta s} K(u^1,u^2)\sqrt{g_{11}}\,du^1du^2 = -\iint_{\delta s}\frac{\partial^2\sqrt{g_{11}}}{\partial(u^2)^2}du^1du^2. \tag{10}$$

Where $dA = \sqrt{g_{11}}du^1du^2$, from (5) and (10) we obtain

$$\Delta(c) = \iint_{\delta s} K_v dA_v = -\iint_{\delta s}\frac{\partial^2\sqrt{g_{11}}}{\partial(u^2)^2}du^1du^2$$

$$= -\oint_c \frac{\partial\sqrt{g_{11}}}{\partial u^2}du^1 = -\oint_c \frac{\partial\sqrt{g_{11}}}{\partial u^2}\dot{u}^1 ds$$

$$= \oint_c \dot{\varphi}\, ds. \tag{11}$$

where ds is the line element. In eguation (11) we have used the eguation of parallel displacement[9–10]

$$\dot{\varphi} = \frac{\partial \sqrt{g_{11}}}{\partial u^2} \dot{u}^1, \tag{12}$$

which suits any coordinate. Eguation (11) shows that the intrinsic angular difference $\Delta(c_\alpha)$ is not relative to the area δA of δs. Therefore, $\Delta(c)$ in (7) and (8) can be considred to be the proportional constant, and which is only relative to the path of the parallel displacement of unite tangent vector $\omega(s)$ aronnd the boundary curve C. That is to say, the proportional constant $\Delta(c)$ in (7) and (8) should be determined by the boundary condition C of area element δs.

B. Boundary Condition and Proportional Constant

i) C is the continuously differentiable closed curve

From rotation index theorem, there must be

$$\Delta(c) = 2\pi - \oint_c kds = \iint_R K_v dA. \tag{13a}$$

ii) C is the boundary curve of geodesic triangle T

There must be

$$\Delta(c) = \sum_{i=1}^{3} \varphi_i - \pi = \iint_T K_v dA = \delta A_g, \tag{13b}$$

Where φ_i is the interior angles of T. Because $\varphi_i < \pi$, thus in (14b) we have

$$\Delta(c) = \delta A_g = \sum_{i=1}^{3} \varphi_i - \pi < 2\pi. \tag{13c}$$

From Gaussian theorem, the angular excess of geodesic triangle is just egual to the area δA_g of its spherical image, which satisfies the definition of Gaussian curvature

$$K_v = \lim_{\delta A_v \to 0} \frac{\delta A_g}{\delta A_v}. \tag{13d}$$

iii) C is the boundary curve of some spin network (geodesic quadrilateral)

There must be

$$\Delta(c) = \delta A_g = \sum_{i=1}^{4} \varphi_i - 2\pi < 2\pi. \tag{13e}$$

We see in (13e) that the angular excess and the area δA_g of spherical image of some spin network δs_v have the same number domain $(0 < \delta A_g < 2\pi)$, which are only relative to the sum $\sum_{i=1}^{4} \varphi_i$ of interior angles, and which are not relative to the quantized area element δA_v of δs_v. Therefore, δA_g can be considered to be the proportional constant of K_v and δA_v. Inserting (14b) into (8), we obtain

$$K_{v_\alpha} = \frac{\Delta(c_\alpha)}{\delta A_{v_\alpha}} = \frac{\delta A_{g_\alpha}}{8\pi r l_p^2 \sqrt{j_\alpha (j_\alpha + 1)}}, \tag{14a}$$

Which shows that the corresponding eigenvalues of Gaussian curvature K_{v_α} and area element δA_{v_α} are proportional to each other, and the proportional constant is just the area δA_{g_α} of the spherical image of δs_{v_α}.

C. The Angle-Preserving Transformation for Spin Network gives the Same Spherical Image

We will consider the state that the proportional constant δA_{g_α} is some given value. We consider some spin network, which is some guantized area element δs_{v_α}, and which have a lot of possible eigenareas δA_{v_α}. The spin network can be considered as a geodesic quadrilateral. From formula (14b),one can conclude that the area of the spherical image on unital sphere of some spin network δs_{v_α} should be only relative to the sum of the interior angles φ_i of this spin network, and which is not relative to the length its each side. Therefore, we may take the angles-preserving transformation for some quantized spin network δs_{v_α}, thus, the sum $\sum_{i=1}^{4}\varphi_i$ of the interior angles φ_i of respective possible area elements δs_{v_α} should be just the same, and the areas of the spherical image on unital sphere of respective δs_{v_α} should be also the same, that is to say

$$\delta A_{g_\alpha} = \delta A_g. \tag{14b}$$

We see in formula(4b) that the area spectrum of some spin network δs_{v_α} should be

$$\delta A_{v_\alpha} = 8\pi r l_p^2 \sqrt{j_\alpha (j_\alpha + 1)}, \tag{4b}$$

Where the eigenareas δA_{v_α} are only relative to the eigenvalues of spin j_α, the quantum jump of spin j_α leads to the quantum transformation of the eigenareas δA_{v_α} of spin network δs_{v_α}. And formula (4b) shows that the eigenareas δA_{v_α} have not related to the magnitudes of each side length and each interior angle of spin network. Therfore, we may take the angle-preserving transformation as following: Changing the lengthes of the respective sides of some spin network, let the spin network Z with eigenarea $\delta A_{\frac{1}{2}}$ become as another spin network W with eigenarea δA_1 or $\delta A_{3/2}$,⋯,but their corresponding interior angles φ_i (i=1,⋯,4) have not changed. Therefore, the areas of the spherical imagles on unital sphere of eigenareas $\delta A_{\frac{1}{2}}, \delta A_1, \delta A_{3/2},$ ⋯are the same $\delta A_{g_\alpha} = \delta A_g$. The angle-preserving tranfornation as mentioned above is just the unified description of the quantized area element δs_{v_α} under the condition $\sum_{i=1}^{4}\varphi_i =$ constant, as shown in figure 1.

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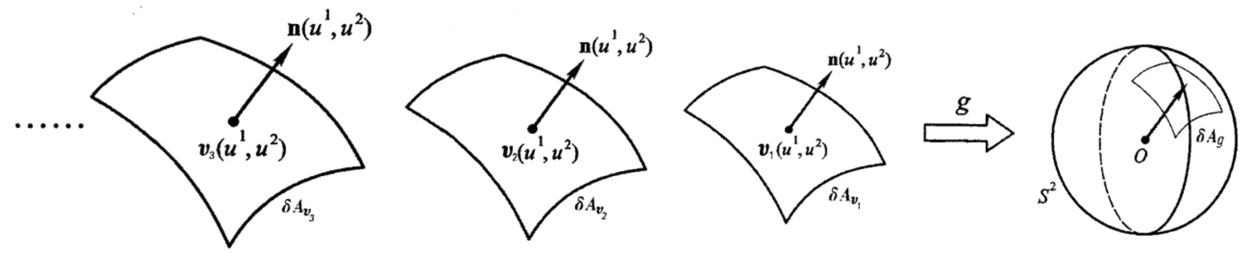

Figure 1 We take the angle-preserving transformations for all the spin networks with different eigenareas δA_{v_α}, which give a lot of similar geodesic quantrilaterals, thus, they have the same spherical image δA_g.

By using this angle-preserving transformation, we can transform the different eigenareas δA_{v_α} with different geometric shapes into a lot of similar geodesic quatrilaterals, they have the same spherical image δA_g, which is just the proportional constant with some given value

$$\delta A_g \in [0, 2\pi]. \tag{15}$$

D. The All Gaussian Curvature of a closed surface at Planck Scale

We suppose that a canonical closed surface consists of many spin networks R_n,and respective boundaries are the canonical closed curves $c_1, \cdots, c_n$, they have positive directions. And the area element of each spin network at Planck scale should be [1]

$$dA = \sqrt{g_s(x)}d^2x = \{8\pi\gamma L_p^2 \delta^{(2)}(x, v_\alpha)\sqrt{j_\alpha(j_\alpha+1)}\}d^2x, \tag{16a}$$

inserting (17a) into the follwing Gaussian-Bonnet theorem in the large

$$\oiint_s dAK = 4\pi, \tag{16b}$$

we obtain

$$\oiint_s dAK = \sum_\alpha \iint d^2x\{8\pi\gamma L_p^2 \delta^{(2)}(x, v_\alpha)\sqrt{j_\alpha(j_\alpha+1)}\}K_{v_\alpha}$$

$$= \sum_\alpha 8\pi\gamma L_p^2 \sqrt{j_\alpha(j_\alpha+1)} \int dx_1 \delta(x_1, v_\alpha) K_{\delta x_1}(x_1) \int dx_2 \delta(x_2, v_\alpha) K_{\delta x_2}(x_2)$$

$$= \sum_\alpha 8\pi\gamma L_p^2 \sqrt{j_\alpha(j_\alpha+1)} \{K_{\delta x_1}(0) \int_{-\delta x_1/2}^{\delta x_1/2} dx_1 \delta(x_1, v_\alpha) \cdot K_{\delta x_2}(0) \int_{-\delta x_2/2}^{\delta x_2/2} dx_2 \delta(x_2, v_\alpha)\}$$

$$= \sum_\alpha 8\pi\gamma L_p^2 \sqrt{j_\alpha(j_\alpha+1)} \{K_{\delta x_1}(0) K_{\delta x_2}(0)\}$$

$$= \sum_\alpha \delta A_{v_\alpha} K_{v_\alpha}(0)$$

$$= \sum_\alpha 8\pi\gamma L_p^2 \sqrt{j_\alpha(j_\alpha+1)} K_{v_\alpha}(0) = \sum_\alpha \Delta(c_\alpha) = \sum_\alpha \delta A_{g_\alpha}$$

$$= 4\pi, \tag{17}$$

where $K_{\delta x_1}(x_1)$ and $K_{\delta x_2}(x_2)$ have no contributions in the integral signs at $x_1 \neq 0$ and $x_2 \neq 0$. Hence, the all Gaussian curvature should be given by

$$_s dAK(x_1, x_2) = \sum_{\alpha} 8 \oint L_p^2 \sqrt{j_\alpha (j_\alpha + 1)} K_{v_\alpha}(0) = 4\pi, \tag{18}$$

which is the quantized Gaussian-Bonnet formula at Planck scale, and which shows that all Gaussian curvature is the geometric invariant. Althongh the sufficiently small area δA_{v_α} of 2-surface and its Gaussian curvature K_{v_α} at the neighborhood of each vertex v_α are quantized, yet the multiplicative product of corresponding eigenvalues δA_{v_α} and K_{v_α} must be egual to some eigenarea δA_{g_α} of spherica limage. (see(8)). And the all Gaussian curvature of a closed spin network (any closed 2-surface, K＞0) must be equal to 4π , although each pair δA_{v_α} and K_{v_α} can take the corresponding different eigenvalues.

The Quantization of Vacuum Einstein Equation at Planck Scale

A. Eigenvalue Equation

The 2-dimention vacuum Einstein equation can be written as [4].

$$R_{1212} = Kg. \tag{19}$$

Now, we rewrite (20) as the eigenvalue equation [5]

$$\hat{R}_{1212} \psi_\gamma = \hat{K}_v \hat{g} \psi_\gamma, \tag{20}$$

which is the quantized vacuum Einstein equation at Planck scale.

It is known that the vacuum and ∧-term solutions of Einstein field equation often admit the subspaces of constant curvature. On a single subspace the Gaussian curvature K is of course constant, but it may have differing values on different subspaces [4].

B.Eigen Curvatures of de Sitter Space

It is known that de Sitter space can be considered to be the solution of vacuum Einstein equation, its metric [4]

$$dS^2 = \frac{dx^2 + dy^2 + dz^2 - dt^2}{[1 + \frac{K}{4}(x^2 + y^2 + z^2 - t^2)]^2}, \tag{21a}$$

which describes the space-time of constant curvature. The metric of the spacelike hypersurface in de Sitter space is

$$dS^2 = \frac{dx^2 + dy^2 + dz^2}{[1 + \frac{K}{4}(x^2 + y^2 + z^2)]^2}, \tag{21b}$$

which has the equivalent form [4]

$$dS^2 = \frac{dr^2}{1 - Kr^2} + r^2 (d\theta^2 + \sin^2\theta d\varphi^2). \tag{21c}$$

Althongh de Sitter space have differing forms in different coordinate systems, yet they should be quantized at Planck scale, because we have proven that Gaussian curvature K_v should be quantized as shown in the formula (8),(13d) and (15), thus we obtain

$$dS^2 = \frac{dx^2 + dy^2 + dz^2}{\left[1 + \frac{1}{4}\left(\frac{\delta A_g}{8\pi\gamma L_p^2\sqrt{j_\alpha(j_\alpha+1)}}\right)(x^2+y^2+z^2)\right]^2}, \tag{22a}$$

And

$$dS^2 = \frac{dr^2}{1 - \left(\frac{\delta A_g}{8\pi\gamma L_p^2\sqrt{j_\alpha(j_\alpha+1)}}\right)r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{22b}$$

We see in (22a) and (22b) that the metric of de Sitter space at Planck scale is not fixed constant curvature metric, but its metric should be quantized at Planck scale, since the Gaussian curvature K_v at Planck scale must be quantized. That is to say, as the solutions of vcuum Einstein equation (eigenvalue equation (21)) at Planck scale are a series of eigen metrics with eigen curvatures.

We should emphasize that r in (21c) and (22b) is not radius distance, and the relation between r and x, y, z is not also the relation of the spherical coordinates and right-angle coordinates.

Discussion

A. Application for Black Holes

We see in (18) that the all Gaussian-curvature is the geometric invariant, when K_{v_α} in (19) are same at the neighborhood of each vertex v_α, (18) becomes

$$K_v \sum_\alpha 8\pi\gamma L_p^2\sqrt{j_\alpha(j_\alpha+1)} = 4\pi, \tag{23}$$

and the total area of a closed 2-surface should be

$$A = \sum_\alpha 8\pi\gamma L_p^2\sqrt{j_\alpha(j_\alpha+1)}. \tag{24}$$

According to our views in this paper, we can explain the quantized relations among the horizon area A^H, Bekenstein-Hawking entropy S^{BH} and Gaussian curvature K^H of A^H. From

$$S^{BH} = \frac{A^H}{4L_p^2} \tag{25}$$

and considering Schwarzschild, Nut-Taub and Kerr-Newman black holes, their horizons have the topology of 2-sphere. Therefore, formula (23) should be suitable for these black holes, thus we have

$$K^H = \frac{4\pi}{A^H}. \tag{26}$$

Comparing (25) and (26), the Gaussian curvature of horizon area can be represented by

$$K^H = \frac{\pi}{S^{BH}L_p^2} \tag{27}$$

We see in (26) and (27) that each eigenvalue of K^H and its corresponding eigenvalue of A^H

must form the geometric invariant; Similarly, each eigenvalue of K^H and its corresponding eigenvalue of S^{BH} also form invariant.

The physical area A^H of horizon is a closed spin network, which is certain finite linear combinations of loop states. Therefore, formulas (26) and (27) show that the increase of A^H and the decrease of K^H should be quantized, the increase of S^{BH} and the decrease of K^H should be also quantized.

B. Why did Gaussian Curvature is Relative to Physical Entropy?

We see in (27) that Gaussian curvature is only relative to physical entropy, that is to say, the physical entropy must influence the curve of surface, why? We would research into this viewpoint at next paper.

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