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Minkowski's type inequality and its functional on time scales

Lifang Liao[1, a] and Xuhuan Wang[2, b]

[1] Institute of Information Technology of Guilin University of Electronic Technology, Guilin, Guangxi, 541004, China

[2] Department of Education Science, Pingxiang College, Pingxiang, Jiangxi, 337055, China

[a] ash90@163.com [b] wangxuhuan84@163.com

Abstract. We establish integral form of the Minkowski's type inequality on time scales. Also, we obtain a converse of Minkowski's type inequality and its functional arising from the Minkowski inequality.

Keywords: Time scale, Minkowski inequality, Minkowski functional..

1. Introduction

In order to unify continuous and discrete analysis. In 1988, Hilger established the theory of time scales in his doctoral dissertation [1] that resulted in his seminal paper [2] in 1990. Since then many authors have studied some integral inequalities on time scales. For example, Wong et al. [3,4] established the delta integral Minkowski's inequality on time scales as follows.

Theorem 1.1 Assume that $f,g,h\in C_{rd}([a,b],\mathbb{R})$ and $1/p+1/q=1$ with $p>1$, then

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^p\Delta x\right)^{\frac{1}{p}}\\&\le\left(\int_a^b |h(x)||f(x)|^p\Delta x\right)^{\frac{1}{p}}+\left(\int_a^b |h(x)||g(x)|^p\Delta x\right)^{\frac{1}{p}}.\end{aligned}\tag{1.1}$$

Ozkan et al. [5] established the nabla and diamond-α integral Minkowski's inequality on time scales which can be stated as follows:

Theorem 1.2 Assume that $f,g,h\in C_{ld}([a,b],\mathbb{R})$ and $1/p+1/q=1$ with $p>1$, then

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^p\nabla x\right)^{\frac{1}{p}}\\&\le\left(\int_a^b |h(x)||f(x)|^p\nabla x\right)^{\frac{1}{p}}+\left(\int_a^b |h(x)||g(x)|^p\nabla x\right)^{\frac{1}{p}}.\end{aligned}\tag{1.2}$$

Theorem 1.3 Assume that $f,g,h:[a,b]\to\mathbb{R}$ are $\Diamond_\alpha$-integrable functions, and $1/p+1/q=1$ with $p>1$, then

$$\begin{aligned}&\left(\int_a^b |h(x)||f(x)+g(x)|^p\Diamond_\alpha x\right)^{\frac{1}{p}}\\&\le\left(\int_a^b |h(x)||f(x)|^p\Diamond_\alpha x\right)^{\frac{1}{p}}+\left(\int_a^b |h(x)||g(x)|^p\Diamond_\alpha x\right)^{\frac{1}{p}}.\end{aligned}\tag{1.3}$$

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Recently, Chen [6] further generalized inequality (1.3) as follows.

Theorem 1.4 Assume that $f,g,h:[a,b]\to\mathbb{R}$ are $\Diamond_\alpha$-integrable functions, $p>0$, $s,t\in R\setminus\{0\}$, and $s\neq t$. Let $p,s,t\in R$ be different, such that $s,t>1$ and $(s-t)/(p-t)>1$, then

$$\int_a^b |h(x)||f(x)+g(x)|^p \Diamond_\alpha x \leq \left[\left(\int_a^b |h(x)||f(x)|^s \Diamond_\alpha x\right)^{\frac{1}{s}} + \left(\int_a^b |h(x)||g(x)|^s \Diamond_\alpha x\right)^{\frac{1}{s}}\right]^{s(p-t)/(s-t)} \times\left[\left(\int_a^b |h(x)||f(x)|^t \Diamond_\alpha x\right)^{\frac{1}{t}} + \left(\int_a^b |h(x)||g(x)|^t \Diamond_\alpha x\right)^{\frac{1}{t}}\right]^{t(p-t)/(s-t)}, \tag{1.4}$$

with equality if and only if the functions $|f|$ and $|g|$ are proportional.

The purpose of this paper is to establish integral form of the Minkowski's type inequality on time scales. Also, we establish a converse of Minkowski's type inequality and its functional arising from the Minkowski inequality. The reader is referred to [1,2,7-9] for an account of the calculus corresponding to the delta derivative, the nabla derivative and diamond-α dynamic derivative, respectively.

A time scale $\mathbb{T}$ means an arbitrary nonempty closed subset of the real numbers. In [10,11], Martin Bohner and Gusein Sh. Useinov established the multiple Riemann and multiple Lebesgue integration on time scales and compared the Lebesgue $\Delta-$integral with the Riemann Δ-integral. For more details, one can see [10,11].

In order to prove our main results, we need the folloing theorems.

Theorem 1.5. (see Conf[12]) Assume that (X,μ,μ_Δ) and $(Y,\mathbb{L},v_\Delta)$ are two finite dimensional time scale measure spaces. If $f:X\times Y\to \mathrm{R}$ is a Δ-integrable function and if we define the functions

$$\varphi(y)=\int_X f(x,y)\,d\mu_\Delta(x) \quad \text{for a.e. } y\in Y$$

and

$$\varphi(x)=\int_Y f(x,y)\,d\mu_\Delta(y) \quad \text{for a.e. } x\in X,$$

then φ is $\Delta-$integrable on Y and ψ is Δ-integrable on X and

$$\int_X d\mu_\Delta(x)\int_Y f(x,y)\,dv_\Delta(y)=\int_Y dv_\Delta(y)\int_X f(x,y)\,d\mu_\Delta(x) \tag{1.5}$$

Theorem 1.6 (see[13]) For $p\neq 1$, let $q=p/(p-1)$. Assume that (E,F,μ_Δ) is a time scale measure space. Assume that w,f,g are nonnegative functions such that wf^p, wg^q, wfg are Δ-integrable on E. If $p>1$, then

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$$\int_E w(t) f(t) g(t) d\mu_\Delta(t) \le \left(\int_E w(t) f^p(t) d\mu_\Delta(t)\right)^{1/p} \left(\int_E w(t) g^q(t) d\mu_\Delta(t)\right)^{1/q}. \tag{1.6}$$

If $0 < p < 1$ and $\int_E wg^q d\mu_\Delta > 0$ or if $p < 0$ and $\int_E wf^p d\mu_\Delta > 0$, then (1.6) is reversed.

2. Main results

Theorem 2.1 Assume that $f(x), g(x) \ge 0$ and $p > 0$ or $f(x), g(x) > 0$ and $p < 0$. Let $s,t \in \mathbb{R} \setminus \{0\}$ and $s,t \ne 0$. Then

(i) Let $s,t \in \mathbb{R}$ be different, such that $s,t > 1$ and $(s-t)/(p-t) > 1$. Then

$$\int_X \left(\int_Y f(x,y) v(y) dv_\Delta(y)\right)^p u(x) d\mu_\Delta(x)$$
$$\le \left[\int_Y \left(\int_X f^s(x,y) u(x) d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y) dv_\Delta(y)\right]^{s(p-t)/(s-t)}$$
$$\times \left[\int_Y \left(\int_X f^t(x,y) u(x) d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y) dv_\Delta(y)\right]^{t(s-p)/(s-t)},$$

with equality if and only if $f(x)$ and $g(x)$ are constant, or $1/p = (1/s + 1/t)/2$ and $f(x)$ and $g(x)$ are proportional..

(ii) Let $s,t \in \mathbb{R}$ be different, such that $s,t < 1$ and $s, t \ne 0$, and $(s-t)/(p-t) < 1$. Then

$$\int_X \left(\int_Y f(x,y) v(y) dv_\Delta(y)\right)^p u(x) d\mu_\Delta(x)$$
$$\le \left[\int_Y \left(\int_X f^s(x,y) u(x) d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y) dv_\Delta(y)\right]^{s(p-t)/(s-t)}$$
$$\times \left[\int_Y \left(\int_X f^t(x,y) u(x) d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y) dv_\Delta(y)\right]^{t(s-p)/(s-t)},$$

with equality if and only if $f(x)$ and $g(x)$ are constant, or $1/p = (1/s + 1/t)/2$ and $f(x)$ and $g(x)$ are proportional.

Proof . we use Minkowski's inequality for $s > 1$ and $t > 1$, respectively, we obtain

$$\int_X \left(\int_Y f(x,y) v(y) dv_\Delta(y)\right)^s u(x) d\mu_\Delta(x)$$
$$\le \left[\int_Y \left(\int_X f^s(x,y) u(x) d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y) dv_\Delta(y)\right]^s$$

and

$$\int_X \left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)^t u(x)d\mu_\Delta(x)$$
$$\le \left[\int_Y \left(\int_X f^t(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^t .$$

Put $H(x)=\int_Y f(x,y)v(y)d\nu_\Delta(y)$, we have $s,\, t>1$ and $(s-t)/(p-t)>1$, by using Fubini's theorem (Theorem 1.5) and H¨older's inequality (Theorem 1.6) on time scales, we have

$$\int_X H^p(x)u(x)d\mu_\Delta(x)$$
$$=\int_X (H^s(x))^{(p-t)/(s-t)} (H^t(x))^{(s-p)/(s-t)} u(x)d\mu_\Delta(x)$$
$$\le \left[\int_X H^s(x)u(x)d\mu_\Delta(x)\right]^{(p-t)/(s-t)} \left[\int_X H^t(x)u(x)d\mu_\Delta(x)\right]^{(s-p)/(s-t)}$$
$$=\left[\int_X \left(\int_Y f^s(x,y)v(y)d\nu_\Delta(y)\right)^{\frac{1}{s}} u(x)d\mu_\Delta(x)\right]^{s(p-t)/(s-t)}$$
$$\times\left[\int_X \left(\int_Y f^t(x,y)v(y)d\nu_\Delta(y)\right)^{\frac{1}{t}} u(x)d\mu_\Delta(x)\right]^{t(s-p)/(s-t)}$$
$$\le \left[\int_Y \left(\int_X f^s(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$
$$\times\left[\int_Y \left(\int_X f^t(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)} .$$

For $p<0$ and $0<p<1$, the corresponding results can be established similarly.

Consider the functional M defined by

$$M(u)=\left[\int_Y \left(\int_X f^s(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$
$$\left[\int_Y \left(\int_X f^t(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}$$
$$-\int_X \left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)^p u(x)d\mu_\Delta(x)$$

provided that all occurring integrals hold.

Theorem 2.2. (i) Assume that $p\ge 1$ *or* $p<0$, then M is superadditive. Assume that $0<p<1$,then M is subadditive.

(iii) Assume that u_1 *510*and u_2 are nonnegative functions such that $u_2\ge u_1$. *Let* $p\ge 1$ *or p* < 0. *then*

$$0\le M(u_1)\le M(u_2).$$

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Proof. First we show (i). We have

$$M_1(u_1+u_2)-M_1(u_1)-M_1(u_2)$$

$$=\left[\int_Y\left(\int_X f^s(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{s}}v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{t}}v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}$$

$$-\left[\int_Y\left(\int_X f^s(x,y)u_1(x)d\mu_\Delta(x)\right)^{\frac{1}{s}}v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)u_1(x)d\mu_\Delta(x)\right)^{\frac{1}{t}}v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}$$

$$-\left[\int_Y\left(\int_X f^s(x,y)u_2(x)d\mu_\Delta(x)\right)^{\frac{1}{s}}v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)u_2(x)d\mu_\Delta(x)\right)^{\frac{1}{t}}v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)},$$

$$M(u_1+u_2)=\left[\int_Y\left(\int_X f^s(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{s}}v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{t}}v(y)d\nu_\Delta(y)\right]$$

$$-\left[\int_Y\left(\int_X f^t(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{t}}v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)},$$

$$M(u_1)=\left[\int_Y\left(\int_X f^s(x,y)u_1(x)d\mu_\Delta(x)\right)^{\frac{1}{s}}v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)u_1(x)d\mu_\Delta(x)\right)^{\frac{1}{t}}v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}$$

$$-\int_X\left(\int_Y f(x,y)v(y)d\nu_\Delta(y)\right)^p u_1(x)d\mu_\Delta(x),$$

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$$M(u_2)=\left[\int_Y\left(\int_X f^s(x,y)(u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)(u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]$$

$$-\left[\int_Y\left(\int_X f^t(x,y)(u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)},$$

$$M(u_1+u_2)-M(u_1)-M(u_2)$$

$$=\left[\int_Y\left(\int_X f^s(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)(u_1+u_2)(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}$$

$$-\left[\int_Y\left(\int_X f^s(x,y)u_1(x)d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)u_1(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}$$

$$-\left[\int_Y\left(\int_X f^s(x,y)u_2(x)d\mu_\Delta(x)\right)^{\frac{1}{s}} v(y)d\nu_\Delta(y)\right]^{s(p-t)/(s-t)}$$

$$\left[\int_Y\left(\int_X f^t(x,y)u_2(x)d\mu_\Delta(x)\right)^{\frac{1}{t}} v(y)d\nu_\Delta(y)\right]^{t(s-p)/(s-t)}.$$

Apping the Minkowski inequality with s,t replaced by $1/s, 1/t$, we have

$$M(u_1+u_2)-M(u_1)-M(u_2)\begin{cases}\geq 0 & if \quad s,t\geq 1\\ \leq 0 & if \quad 0<s,t\leq 1.\end{cases}$$

So, M is superadditive for $s,t\geq 1$, and it is subadditive for $0<s,t\leq 1$.

If $s,t\geq 1$, then using superadditivity and positivity of M, $u_1>u_2$ implies

$$M(u_2)=M((u_1+(u_2-u_1))\geq M(u_1)+M(u_1+u_2)>M(u_1).$$

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