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Positive solution of nonlinear two-order three-point boundary value problem for difference equation with change of sign

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Abstract. In this paper we investigate the existence of positive solution of the following discrete two-order three-point boundary value problem.

$\Delta^2 y_{t-1} + \lambda a(t) f(y_t) = 0, t \in [1,n],$

$y_0 = 0, by_p = y_{n+1},$

where λ is a positive parameter, $n \in [2,+\infty)$, $p \in [2,n], 0 < b < 1, bp \in [1, p-1]$ and $f \in C(R^+, R^+)$. We will use the Krasnoselskii's fixed-point theorem and obtain the existence of positive solution of the boundary value problem in a cone, here $a(t)$ is allowed to change sign on $[1,n]$. An example is given to demonstrate the applications of the theorems obtained.

Introduction

In this paper, we will consider the existence of positive solution for the nonlinear discrete three-point boundary value problem

$$\Delta^2 y_{t-1} + \lambda a(t) f(y_t) = 0, \ t \in [1,n], \tag{1}$$

$$y_0 = 0, by_p = y_{n+1}, \tag{2}$$

where λ is a positive parameter, $n \in [2,+\infty)$, $p \in [2,n], 0 < b < 1, bp \in [1, p-1]$ and $f \in C(R^+, R^+)$. Recently, some authors considered the existence and uniqueness of positive solutions of discrete boundary value problems (See[1,8,10,11]) and obtained some existence results. In [12], G.Zhang studied the existence and nonexistence of the following discrete three-point boundary value problem

$$\Delta^2 x_{k-1} + f(x_t) = 0, \ k = 1,2,...,n. \tag{3}$$

We want to point out that our problem is different from [12]. Moreover, to the author's knowledge, no one has studied the existence of positive solution for problem (1), (2) using the assumptions that $a(t)$ is allowed to change sign on $[1,n]$.

Hence, we will establish some criteria for the existence of at least one positive solution of BVP(1), (2). In addition, the following Krasnoselskii's fixed-point theorem is the key tool in our approach.

Theorem 1.1.(See[7, 9]) Let E be Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$, and let $A: K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator. In addition, suppose either $\| Au \| \le \| u \|, \ for \ u \in K \cap \partial\Omega_1$ and $\| Au \| \ge \| u \|, \ for \ u \in K \cap \partial\Omega_2 \ or \ \| Au \| \ge \| u \|,$

for $u \in K \cap \partial\Omega_1$, *and* $\| Au \| \le \| u \|$, *for* $u \in K \cap \partial\Omega_2$ holds. Then A has a fixed point in $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

In this paper, a positive solution y^* of the BVP(1), (2) will mean a solution y^* of the BVP(1), (2) satisfying $y_t^* > 0, 0 < t < n+1$. Moreover, we shall use the following assumptions.

(H_1) $f \in C(R^+, R^+)$ is continuous and nondecreasing.

(H_2) $a:[1,n] \to (-\infty, +\infty)$ is continuous and such that $a(t) \ge 0, t \in [1,p]; a(t) \le 0$, $t \in [p,n]$. Moreover, $a(t)$ doesn't vanish identically on any subinterval of $[1,n]$.

(H_3) There exist nonnegative constants in the extended reals, f_0, f_∞, such that

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u},\ f_\infty = \lim_{u \to +\infty} \frac{f(u)}{u}.$$

(H_4) For all $t \in [0, n-p]$, there exists a constant $\tau \in (bp, p)$ such that the function

$$A(t) = a^+(p - \delta t) - \frac{1}{\Lambda} a^-(p+t) \ge 0, \quad \text{where}$$

$$a^+(t) = \max\{a(t), 0\}, a^-(t) = -\min\{a(t), 0\},\ \delta = \frac{p-\tau}{n-p}, \text{ and } \Lambda = \frac{b^2 p}{n}.$$

Next, for the sake of convenience, set

$$\alpha(t) = \begin{cases} \dfrac{t}{p}, & 0 \le t \le p, \\ \dfrac{n+1-bp+(b-1)t}{n+1-p}, & p < t \le n+1, \end{cases} \qquad k(t,s) = \begin{cases} k_1(t,s), & 0 \le t \le p, \\ k_2(t,s), & p < t \le n+1, \end{cases} \quad \text{and}$$

$$k_1(t,s) = \begin{cases} k_{11}(t,s) = \dfrac{(b-1)t + (n+1-bp)}{n+1-bp} s, & 0 \le s < t \le p \le n+1, \\ k_{12}(t,s) = \dfrac{(b-1)s + (n+1-bp)}{n+1-bp} t, & 0 \le t \le s < p \le n+1, \\ k_{13}(t,s) = \dfrac{(n+1-p)}{n+1-bp} t, & 0 \le t \le p \le s \le n+1, \end{cases}$$

$$k_2(t,s) \begin{cases} k_{21}(t,s) = \dfrac{(b-1)t + (n+1-bp)}{n+1-bp} s, & 0 \le s < p \le t \le n+1, \\ k_{22}(t,s) = \dfrac{bp(t-s) + (n+1-t)}{n+1-bp} s, & 0 \le p \le s < t \le n+1, \\ k_{23}(t,s) = \dfrac{(n+1-s)}{n+1-bp} t, & 0 \le p \le t < s \le n+1. \end{cases}$$

It is easy to know that $k(t,s) \ge 0, (t,s) \in [0,n+1] \times [0,n+1]$. Moreover, we state and prove an inequality for $k(t,s)$ (see Lemma2.4 in Section 2). In addition, set

$$\Lambda_1 = \mu \max_{t \in [0,n+1]} \sum_{s=bp}^{\tau} k(t,s) a^+(s),\ \Lambda_2 = \max_{t \in [0,n+1]} \sum_{s=1}^{p} k(t,s) a^+(s), \text{where} \quad \mu = \min\{b, 1 - \frac{\tau}{p}\}.$$

The paper is organized as follows. In next section, we present some notations and preliminaries. The main results, existence of positive solution of BVP (1), (2) is given in Section 3. In section 4, we will give an example to illustrate our main results.

Preliminaries and Lemmas

Throughout this paper, we always use the following notations and signs, $Z = \{0, \pm 1, \pm 2, ...\}$; $N = \{0, 1, 2, ...\}$; $[m,n] = \{m, m+1, m+2, ...\} \subset Z$; $\forall x \in R$, $[x]$ is the integer value function; $\Delta y_t = y_{t+1} - y_t, \Delta^n y_t = \Delta(\Delta^{n-1} y_t), n \geq 2, t \in N$. In order to discuss problem (1), (2), the preliminary lemmas are in this section. Now, let $C[0, n+1]$ be the Banach space with norm $\| y \| = \sup_{t \in [0, n+1]} | y_t |$. Denote

$$C_0^+[0, n+1] = \{ y_t \in C[0, n+1] : \min_{t \in [0, n+1]} y_t \geq 0 \, and \; y_0 = 0, y_{n+1} = b y_p \},$$

$$P = \{ y_t \in C_0^+[0, n+1]: y_t \text{ is concave on} [0, p], \text{ and convex on } [p, n+1] \}.$$

It is obvious that P is a cone in $C[0, n+1]$.

Lemma 2.1. Let $y_t \in P$, then $y_t \geq \alpha(t) y_p, t \in [0, p]$, and $y_t \leq \alpha(t) y_p, t \in [p, n+1]$.

Lemma 2.2. Let $y_t \in P$, then $y_t \geq \mu \| y \|, t \in [bp, \tau]$.

Lemma 2.3. (See[12]) Let $bp \neq n+1$. Then for $h_t \in C[1, n]$, the problem $\Delta^2 y_{t-1} + h_t = 0,\; t \in [1, n], y_0 = 0, b y_p = y_{n+1}$, has a unique solution

$$y_t = \frac{t}{n+1-bp} \left(\sum_{i=0}^{n} \sum_{j=0}^{i} h_j - b \sum_{i=0}^{p-1} \sum_{j=0}^{i} h_j \right) - \sum_{i=0}^{t-1} \sum_{j=0}^{i} h_j, t \in [0, n+1], \text{ this is } y_t = \sum_{s=1}^{n} k(t,s) h_s.$$

Lemma 2.4. For all $s_1 \in [\tau, p], s_2 \in [p, n]$, then

$$k(t, s_1) \geq \Lambda k(t, s_2), t \in [0, n+1]. \tag{4}$$

Lemma 2.5. Let conditions $(H_1), (H_2)$ and (H_4) hold. Then, for all $q \in [0, +\infty)$,

$$\sum_{s=\tau}^{p} k(t,s) a^+(s) f(q\alpha(s)) \geq \sum_{s=p}^{n} k(t,s) a^-(s) f(q\alpha(s)).$$

Proof. By the definition of $\alpha(t)$, for each $r \in [0, n-p]$, it is easy to get $\alpha(p - \frac{p-\tau}{n-p} r) = 1 - \frac{r}{n-p} \cdot (1 - \frac{\tau}{p})$, and $\alpha(p+r) = \frac{n-p-r}{n-p}(1-b)$. Thus, in view of $\tau \in (bp, p)$ and f is nondecreasing, for $r \in [0, n-p]$, and $\delta = \frac{p-\tau}{n-p}$, we have $f(1 - \frac{r}{n-p}(1-\frac{\tau}{p})) \geq f(1 - \frac{r}{n-p}(1-b))$. Set $s = p - \delta r, r \in [0, n-p]$, for all $q \in [0, \infty)$, by view of Lemma 2.4 and condition (H_4), we obtain

$$\begin{aligned}
\sum_{s=\tau}^{p} k(t,s) a^+(t) f(q\alpha(s)) &= \sum_{r=0}^{n-p} k(t, p-\delta r) a^+(p-\delta r) f(q\alpha(p-\delta r)) \\
&= \sum_{r=1}^{n-p} k(t, p-\delta r) a^+(p-\delta r) f(q(1 - \frac{r}{n-p}(1-\frac{\tau}{p}))) \\
&\geq \Lambda \sum_{r=1}^{n-p} k(t, p+r) a^+(p-\delta r) f(q(1 - \frac{r}{n-p}(1-\frac{\tau}{p}))) \\
&\geq \sum_{r=1}^{n-p} k(t, p+r) a^-(p+r) f(q(1 - \frac{r}{n-p}(1-b))).
\end{aligned}$$

Again, setting $s = p + r, r \in [0, n-p]$, for $q \in [0, \infty)$, we get

$$\sum_{s=p}^{n} k(t,s) a^-(s) f(q\alpha(s)) = \sum_{r=1}^{n-p} k(t, p+r) a^-(p+r) f(q(1 - \frac{r}{n-p}(1-b))).$$

The proof is completed.

Now we define an operator $T: P \to P$ by

$$(Ty)_t = \lambda \sum_{s=1}^{n} k(t,s) a(s) f(y_s), (t,s) \in [0, n+1] \times [1, n]. \tag{5}$$

Lemma 2.6. Assume that conditions $(H_1), (H_2)$ and (H_4) are satisfied. Then $T: P \to P$ is completely continuous.

Proof. At first, we show that $T: P \to P$. For $y \in P$, by Lemma 2.1, Lemma 2.5, and f is nondecreasing, we have

$$\sum_{s=\tau}^{n} k(t,s)a(s)f(y_s) = \sum_{s=\tau}^{p} k(t,s)a^{+}(s)f(y_s) - \sum_{s=p}^{n} k(t,s)a^{-}(s)f(y_s)$$
$$\geq \sum_{s=\tau}^{p} k(t,s)a^{+}(s)f(\alpha(s)y_p) - \sum_{s=p}^{n} k(t,s)a^{-}(s)f(\alpha(s)y_p) \geq 0,$$

which implies

$$(Ty)_t = \lambda \sum_{s=1}^{n} k(t,s)a(s)f(y_s) = \lambda \sum_{s=1}^{n} k(t,s)a^{+}(s)f(y_s) + \lambda \sum_{s=\tau}^{n} k(t,s)a(s)f(y_s)$$
$$\geq \lambda \sum_{s=1}^{\tau} k(t,s)a^{+}(s)f(y_s) \geq 0,$$

again $(Ty)_0 = 0, (Ty)_{n+1} = b(Ty)_p$, it follows That $T: P \to C_0^{+}[0, n+1]$. On the other hand,

$$\Delta^2 (Ty)_t = -\lambda a^{+}(s)f(y_s) \leq 0, s \in [0,p],\ \Delta^2 (Ty)_t = \lambda a^{-}(s)f(y_s) \geq 0, s \in [p, n+1].$$

Thus, $T: P \to P$. Next, it is easy to prove that $T: P \to P$ is completely continuous by the Arzela-Ascoli theorem. The proof is completed.

Lemma 2.7. Assume that conditions (H_1), (H_2) and (H_4) are satisfied. If $y^* \in P$ is a fixed point of T and $\| y^* \| \geq 0$, then y^* is a positive solution of the BVP (1), (2).

Proof. At first, we claim that $y_p^* > 0$. Otherwise, $y_p^* = 0$, which implies $y_{n+1}^* = by_p^* = 0$. Since y_p^* is nonnegative convex function on $[p, n+1]$, we have $y_t^* \equiv 0, t \in [p, n+1]$, it implies $\Delta y_p^* = y_{p+1}^* - y_p^* = 0$. Again, since $y^* = Ty^*$ and $\Delta^2 y_t^* = -\lambda a^{+}(s)f(y_s) \leq 0$, $s \in [0, p]$, then $\Delta y_t^* \geq \Delta y_p^* = 0, t \in [0, p-1]$. Thus, we get $y_t^* \leq y_p^* = 0, t \in [0, p]$. Since y_p^* is nonnegative function, hence $y_t^* \equiv 0, t \in [0, p]$, which yields contradiction with $\| y_p^* \| \geq 0$.

Next, in view of Lemma 2.1, for $y_t^* \in P$, we have

$$y_t^* \geq \frac{t}{p} y_p^* > 0,\ t \in (0, p]. \tag{6}$$

Notice that $a(t)$ does not vanish identically on any subinterval of $t \in [1, p]$, by the proof of Lemma 2.6 and as shown in Eq.6, for any $t \in (0, n+1)$, we have

$$y_t^* = (Ty^*)_t = \lambda \sum_{s=1}^{n} k(t,s)a(s)f(y_s^*) = \lambda \sum_{s=1}^{\tau} k(t,s)a^{+}(s)f(y_s^*) + \lambda \sum_{s=\tau}^{n} k(t,s)a(s)f(y_s^*)$$
$$\geq \lambda \sum_{s=1}^{\tau} k(t,s)a^{+}(s)f(y_s^*) > 0.$$

Thus, we assert that y^* is a positive solution of the BVP (1), (2).

Existence of Solution

Theorem 3.1. Assume that conditions $(H_1) \sim (H_4)$ are satisfied, $\Lambda_2 f_0 < \Lambda_1 f_\infty$. Then for all λ satisfying

$$\frac{1}{\Lambda_1 f_\infty} < \lambda < \frac{1}{\Lambda_2 f_0}, \tag{7}$$

there exists at least one positive solution of the BVP(1), (2), which belongs to P. Moreover, in the case where f is superlinear, then as shown in Eq.7 becomes $0<\lambda<\infty$.

Proof. Let $\varepsilon>0$ such that $0<\lambda \le 1/\Lambda_2(f_0+\varepsilon)$. Since $f_0=\lim_{u\to 0^+}(f(u)/u)$, there exists $\rho_*>0$ such that

$$f(u)\le (f_0+\varepsilon)u,\ for\ u\in[0,\rho_*]. \tag{8}$$

Let $\Omega_{\rho_*}=\{y\in P:\|y\|<\rho_*\}$. For $y\in\partial\Omega_{\rho_*}\subset P$, as shown in Eq.5.,8., we have

$$(Ty)_t=\lambda\sum_{s=1}^{n}k(t,s)a(s)f(y_s)=\lambda\sum_{s=1}^{p}k(t,s)a^+(s)f(y_s)-\lambda\sum_{s=p}^{n}k(t,s)a^-(s)f(y_s)$$
$$\le\lambda\sum_{s=1}^{p}k(t,s)a^+(s)f(y_s)\le\lambda\Lambda_2(f_0+\varepsilon)\rho_*\le\rho_*=\|y\|,$$

which yields

$$\|Ty\|\le\|y\|,\ for\ \ y\in P\bigcap\partial\Omega_{\rho_*}. \tag{9}$$

Now, we consider two cases.

CASE 1. If $f_\infty<\infty$, set $\varepsilon_1>0$ such that $0<\dfrac{1}{\Lambda_1(f_\infty-\varepsilon_1)}\le\lambda$. Since $f_\infty=\lim\limits_{u\to\infty}(f(u)/u)$, there exist $\rho^*>\rho_*$ such that

$$f(u)\ge(f_\infty-\varepsilon_1)u,\ for\ u\ge\mu\rho^*. \tag{10}$$

Set $\Omega_{\rho^*}=\{y\in P:\|y\|<\rho^*\}$. For any $y\in\partial\Omega_{\rho^*}\subset P$, from Lemma (2.2), we have $y_t\ge\mu\|y\|=\mu\rho^*$, for $t\in[bp,\tau]$. As shown in Eq.5.,10., we obtain

$$\|Ty\|=\lambda\max_{t\in[0,n+1]}[\sum_{s=1}^{\tau}k(t,s)a^+(s)f(y_s)+\sum_{s=\tau}^{n}k(t,s)a(s)f(y_s)]$$
$$\ge\lambda\max_{t\in[0,n+1]}\sum_{s=1}^{\tau}k(t,s)a^+(s)f(y_s)\ge\lambda\max_{t\in[0,n+1]}\sum_{s=bp}^{\tau}k(t,s)a^+(s)f(y_s)$$
$$\ge\lambda\mu(f_\infty-\varepsilon_1)\rho^*\max_{t\in[0,n+1]}\sum_{s=bp}^{\tau}k(t,s)a^+(s)=\lambda\Lambda_1(f_\infty-\varepsilon_1)\rho^*\ge\rho^*=\|y\|,$$ this is

$$\|Ty\|\ge\|y\|,\ for\ y\in P\bigcap\partial\Omega_{\rho^*}. \tag{11}$$

Therefore, as shown in Eq.9.,11. and the Theorem 1.1, it follows that T has a fixed point in $P\bigcap(\bar{\Omega}_{\rho^*}\setminus\Omega_{\rho_*})$. Hence, from Lemma 2.7, there exists at least one positive solution of the BVP(1), (2).

CASE 2. If $f_\infty=\infty$, let $\lambda>0$, take $M>0$ such that $\lambda\Lambda_1 M\ge 1$. Since $\lim\limits_{u\to\infty}(f(u)/u)=\infty$, there exists some ρ^* satisfying $\rho^*>\rho_*$ such that $f(u)\ge Mu, u\ge\mu\rho^*$. Set $\Omega_{\rho^*}=\{y\in P:\|y\|<\rho^*\}$. As before, we get

$$\|Ty\|=\lambda\max_{t\in[0,n+1]}[\sum_{s=1}^{\tau}k(t,s)a^+(s)f(y_s)+\sum_{s=\tau}^{n}k(t,s)a(s)f(y_s)]$$
$$\ge\lambda\mu M\rho^*\max_{t\in[0,n+1]}\sum_{s=bp}^{\tau}k(t,s)a^+(s)f(y_s)=\lambda\Lambda_1 M\rho^*\ge\rho^*=\|y\|,$$ and the proof proceeds as before. The proof of Theorem 3.1 is completed.

Theorem 3.2. Assume that conditions $(H_1)\sim(H_4)$ are satisfied, and $\Lambda_2 f_\infty<\Lambda_1 f_0$. Then for all λ satisfying

$$\frac{1}{\Lambda_1 f_0} < \lambda < \frac{1}{\Lambda_2 f_\infty}, \tag{12}$$

there exists at least one positive solution of the BVP(1), (2), which belongs to P. Moreover, in the case where f is sublinear, then as shown in Eq.12. becomes $0 < \lambda < \infty$.

Proof. At first, for $f_0 = \lim_{u\to 0^+}(f(u)/u)$, we consider two cases.

CASE 1. If $f_0 < \infty$. Set $\varepsilon_1 > 0$ such that $0 < 1/\Lambda_1(f_0 - \varepsilon_1) \le \lambda$, Since $f_0 = \lim\limits_{u\to 0^+}(f(u)/u)$, there exists a positive constant ρ_* such that

$$f(u) \ge (f_0 - \varepsilon_1)u, \ for\ 0 < u < \rho_*. \tag{13}$$

Set $\Omega_{\rho_*} = \{y \in P : \|y\| < \rho_*\}$. For any $y \in \partial\Omega_{\rho_*} \cap P$, by Lemma 2.2, we get $y_t \ge \mu\|y\| = \mu\rho^*$, for $t \in [bp, \tau]$, then as shown in Eq.5.,13., we have

$$\begin{aligned}\|Ty\| &= \lambda \max_{t\in[0,n+1]}[\sum_{s=1}^{\tau} k(t,s)a^+(s)f(y_s) + \sum_{s=\tau}^{n} k(t,s)a(s)f(y_s)] \\ &\ge \lambda \max_{t\in[0,n+1]}\sum_{s=1}^{\tau} k(t,s)a^+(s)f(y_s) \ge \lambda \max_{t\in[0,n+1]}\sum_{s=bp}^{\tau} k(t,s)a^+(s)f(y_s) \\ &\ge \lambda\mu(f_0 - \varepsilon_1)\rho_* \max_{t\in[0,n+1]}\sum_{s=bp}^{\tau} k(t,s)a^+(s) = \lambda\Lambda_1(f_0 - \varepsilon_1)\rho_* \ge \rho_* = \|y\|,\end{aligned}$$

this is

$$\|Tu\| \ge \|u\|, \ for\ y \in P \cap \partial\Omega_{\rho_*}. \tag{14}$$

CASE 2. If $f_0 = \infty$, let $\lambda > 0$, and take $M > 0$ such that $\lambda\Lambda_1 M \ge 1$. Since $\lim\limits_{u\to 0^+}(f(u)/u) = \infty$, there exists a positive constant ρ_* such that $f(u) \ge Mu, \ for\ 0 < u <$ ρ_*. Set $\Omega_{\rho^*} = \{y \in P : \|y\| < \rho^*\}$. As before, we have

$$\begin{aligned}\|Ty\| &= \lambda \max_{t\in[0,n+1]}[\sum_{s=1}^{\tau} k(t,s)a^+(s)f(y_s) + \sum_{s=\tau}^{n} k(t,s)a(s)f(y_s)] \\ &\ge \lambda\mu M\rho_* \max_{t\in[0,n+1]}\sum_{s=bp}^{\tau} k(t,s)a^+(s)f(y_s) = \lambda\Lambda_1 M\rho_* \ge \rho_* = \|y\|,\end{aligned}$$

hence, Eq.14. holds too.

Next, let $\varepsilon > 0$ such that $0 < \lambda \le 1/\Lambda_2(f_\infty + \varepsilon)$. Since $f_\infty = \lim_{u\to\infty}(f(u)/u)$, there exists some ρ satisfying $\rho > \rho_*$ such that

$$f(u) \le (f_\infty + \varepsilon)u, \ for\ u \ge \rho. \tag{15}$$

Now, there are two cases to consider too. They are f is unbounded and bounded. If f is unbounded. Since f is continuous, we know that there is a positive constant ρ^* and $\rho^* > \rho$ such that

$$f(u) \le f(\rho^*), \ for\ u \in [0, \rho^*]. \tag{16}$$

Since $\rho^* > \rho$, then as shown in Eq.15.,16., we obtain

$$f(u) \le f(\rho^*) \le (f_\infty + \varepsilon)\rho^*, u \in [0, \rho^*]. \tag{17}$$

For $y \in P, \|y\| = \rho^*$, from as shown in Eq.4.,17., we have

$$\begin{aligned}(Ty)_t &= \lambda\sum_{s=1}^{n} k(t,s)a(s)f(y_s) = \lambda\sum_{s=1}^{p} k(t,s)a^+(s)f(y_s) - \lambda\sum_{s=p}^{n} k(t,s)a^-(s)f(y_s) \\ &\le \lambda\sum_{s=1}^{p} k(t,s)a^+(s)f(y_s) \le \lambda\Lambda_2(f_\infty + \varepsilon)\rho^* \le \rho^* = \|y\|.\end{aligned}$$

If f is bounded, then there exists a positive constant L_1 such that $f(u) \le L_1$. In the case $f_\infty = 0$. Take $\rho^* \ge \max\{L_1/\varepsilon, \rho\}$, for $y \in P$, $\|y\| = \rho^*$, as shown in Eq.5, one has

$$(Ty)_t = \lambda \sum_{s=1}^{n} k(t,s)a(s)f(y_s) = \lambda \sum_{s=1}^{p} k(t,s)a^+(s)f(y_s) - \lambda \sum_{s=p}^{n} k(t,s)a^-(s)f(y_s)$$

$$\le \lambda \sum_{s=1}^{p} k(t,s)a^+(s)f(y_s) \le \lambda\Lambda_2 L_1 \le \lambda\Lambda_2\varepsilon\rho^* \le \rho^* = \|y\|.$$

Hence, in either case, setting $\Omega_{\rho^*} = \{y \in P : \|y\| < \rho^*\}$, we always have

$$\|Ty\| \le \|y\|, \ for\ y \in P \cap \partial\Omega_{\rho^*}. \tag{18}$$

Therefore, as shown in Eq.14., 18., and Theorem 1.1, it follows that T has a fixed point in $P \cap (\bar{\Omega}_{\rho^*} \setminus \Omega_{\rho_*})$. Hence, from Lemma 2.7, there exists at least one positive solution of the BVP (1), (2). The proof of Theorem 3.2 is completed.

Example

In this section, we will give an example is to illustrate our main results.

Example 4.1 . Consider the boundary value problem

$$\begin{cases} \Delta^2 y_{t-1} + \dfrac{2}{3}a(t)f(y_t) = 0, \ t \in [1,10], \\ y_0 = 0, \qquad \dfrac{1}{2}y_6 = y_{11}, \end{cases} \tag{19}$$

where $n = 10,\ p = 6, b = \frac{1}{2}, \lambda = \frac{2}{3}$, and $a(t) = \begin{cases} 20(t-6)^2, 0 \le t \le 6, \\ \frac{3}{20}(6-t)^3, 6 < t \le 11, \end{cases}$ then we have $bp = 3, \Lambda = \frac{3}{20}$.

Now taking $\tau = 4$, we get $\delta = \frac{1}{2}$. Let

$$y_t = \begin{cases} -(t-5)^2 + 25, \quad 0 \le t \le 6, \\ \dfrac{12}{25}(t-11)^2 + 12, 6 < t \le 11, \end{cases} \text{ and } \ f(u) = \frac{22u^2}{1+2625u} + \frac{2u}{2625}.$$

Then, by calculation, we have

$f_0 = \frac{2}{2625}$, $f_\infty = \frac{8}{875}$, $\Lambda_1 = \frac{825}{4}$, $\Lambda_2 = \frac{2625}{2}$, thus, we have $1 = \Lambda_2 f_0 < \Lambda_1 f_\infty = \frac{66}{35}$, and $\frac{66}{35} = \frac{1}{\Lambda_1 f_\infty} < \lambda = \frac{2}{3} < \frac{1}{\Lambda_2 f_0} = 1$. By Theorem 3.1, we have the BVP (19) has at least one positive solution.

Summary

Assume that conditions $(H_1) \sim (H_4)$ are satisfied. When $\Lambda_2 f_0 < \Lambda_1 f_\infty$, then for all λ satisfying $\frac{1}{\Lambda_1 f_\infty} < \lambda < \frac{1}{\Lambda_2 f_0}$, there exists at least one positive solution of the BVP(1), (2), which belongs to P. Moreover, in the case where f is superlinear, then as shown in Eq.7 becomes $0 < \lambda < \infty$; When $\Lambda_2 f_\infty < \Lambda_1 f_0$, then for all λ satisfying $\frac{1}{\Lambda_1 f_0} < \lambda < \frac{1}{\Lambda_2 f_\infty}$, there exists at least one positive solution of the BVP(1), (2), which belongs to P. Moreover, in the case where f is sublinear, then as shown in

Eq.12. becomes $0 < \lambda < \infty$ are obtained.

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